Also $\sin nx$ is an odd fun^c $\Rightarrow$ $f(x)\sin nx$ will be Even fun^c.

$$\therefore\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx.$$

From above calculation if $f(x)$ is an odd function then Fourier series/Expansion contains only sine terms.

As

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$f(x) \sim 0 + \sum_{n=1}^{\infty}(0 + b_n\sin nx)$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty} b_n\sin nx$$

$$\text{i.e. } b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx \quad \Big|\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

lly when $f(x)$ is an Even function.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$\Rightarrow \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx = \frac{a_0}{2}$$

As $\cos nx$ is an even function

$\Rightarrow f(x)\cos nx$ is also even functions.

$$\therefore \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx.$$

As $\sin nx$ is an odd function.

$\Rightarrow f(x)\sin nx$ will be odd function.

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx = 0$$

From above discussion we have when '$f(x)$' is an Even func Fourier Expansions contains only Cosine terms.

$$\therefore \quad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Theorem :- Fourier Series and Fourier Coefficients theorem

Let the series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$ be uniformly convergent in the closed interval $[\alpha, \alpha+2\pi]$ / $[-\pi, \pi]$ / $[0, \pi]$ and let this series have sum $f(x)$ then.

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\, dx \quad \text{for } n = 0, 1, 2, 3, \ldots$$

and $b_n = \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx$ for $n = 0, 1, 2, 3, \ldots$

Pf:- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ ——— ①

$\forall \; x \in [\alpha, \alpha + 2\pi]$

Since series ① is uniformly convergent in $[\alpha, \alpha+2\pi]$

$\therefore$ we can write '$f(x)$' in integration within $[\alpha, \alpha+2\pi]$

Taking integration w.r.t 'x' within $[\alpha, \alpha+2\pi]$ on ①, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x) dx = \int_{\alpha}^{\alpha+2\pi} \frac{a_0}{2} dx + \int_{\alpha}^{\alpha+2\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) dx.$$

$$= \frac{a_0}{2} \int_{\alpha}^{\alpha+2\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{\alpha}^{\alpha+2\pi} \cos nx \, dx + b_n \int_{\alpha}^{\alpha+2\pi} \sin nx \, dx \right)$$

$$= \frac{a_0}{2} [x]_{\alpha}^{\alpha+2\pi} + 0$$

$\because \int_{\alpha}^{\alpha+2\pi} \cos nx \, dx = 0$

$\int_{\alpha}^{\alpha+2\pi} \sin nx \, dx = 0$

$$= \frac{a_0}{2} [\alpha + 2\pi - \alpha]$$

$$= \frac{a_0}{\cancel{2}} \times \cancel{2}\pi$$

$$= a_0 \pi$$

$$\therefore \boxed{a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx} \quad \text{——— (2)}$$

Now as series (1) is uniformly convergent.

$\therefore$ For given $\epsilon > 0$ $\exists$ a +ve integer n_0 such that

$$|f(x) - S_n(x)| < \epsilon \quad \forall\, n \geq n_0$$

; $S_n(x)$ = nth partial sum of series (1) ——— (3)

consider $\cancel{(f(x) - S_n(x))} =$

$|(f(x)\cos nx - S_n(x)\cos nx)|$

$= |(f(x) - S_n(x))\cos nx| = |f(x) - S_n(x)|\,|\cos nx|$

$\leq |f(x) - S_n(x)| < \epsilon \quad \forall\, n \geq n_0$

$\boxed{\because |\cos nx| \leq 1 \text{ By eq}^n \text{ (3)}}$

$\therefore |f(x)\cos nx - S_n(x)\cos nx| < \epsilon \quad \forall\, n \geq n_0$

$\text{lly} \quad |f(x)\sin nx - S_n(x)\sin nx| < \epsilon \quad \forall\, n \geq n_0$

If we multiply eq^n (1) by $\cos nx$ & $\sin nx$ then ~~obt~~, we get resulting series is also convergent and integrating obtained result between limit α to $\alpha + 2\pi$, we get

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$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi}\cos nx\,dx + \sum_{m=1}^{\infty}\Big[a_m\int_{\alpha}^{\alpha+2\pi}\cos mx\cos nx\,dx$$

$$+ b_m\int_{\alpha}^{\alpha+2\pi}\sin mx\cos nx\,dx\Big]$$

$$= \frac{a_0}{2}[0] + a_m\int_{\alpha}^{\alpha+2\pi}\cos nx\cos nx$$

$$\boxed{\because \int_{\alpha}^{\alpha+2\pi}\cos nx\,dx = 0 \quad n \neq 0 \\ \& \int_{\alpha}^{\alpha+2\pi}\cos mx\sin nx\,dx = 0,\ n \neq m.}$$

$$= a_n\int_{\alpha}^{\alpha+2\pi}\cos^2 nx\,dx = a_n\cdot\pi$$

$$\boxed{\because \int_{\alpha}^{\alpha+2\pi}\cos^2 nx\,dx = \pi}$$

$$\therefore \boxed{a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx} \quad (A)$$

Ily

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi}\sin nx\,dx + \sum_{m=1}^{\infty}\Big[a_m\int_{\alpha}^{\alpha+2\pi}\cos mx\sin nx\,dx$$

$$+ b_m\int_{\alpha}^{\alpha+2\pi}\sin mx\sin nx\,dx\Big]$$

$$= b_n\int_{\alpha}^{\alpha+2\pi}\sin nx\sin nx\,dx = b_n\int_{\alpha}^{\alpha+2\pi}\sin^2 nx\,dx$$

$$\Rightarrow \int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = b_n\pi$$

$$\therefore \boxed{b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx} \text{ ——— } (**)$$

(*) & (**) known as Euler's Formula.

Theorem 2. :- If $f(x)$ is bounded and integrable on $[-\pi, \pi]$ and a_n, b_n the Fourier coefficients then $\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ converges

Pf:- Since $\int_{-\pi}^{\pi}\left[f(x) - \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right]^2 \geq 0$

$$\Rightarrow \int_{-\pi}^{\pi} f^2(x)\,dx + \int_{-\pi}^{\pi}\left[\lim_{\lambda\to\infty}\sum_{n=1}^{\lambda}(a_n\cos nx + b_n\sin nx)\right]^2 dx$$

$$-2\lim_{\lambda\to\infty}\sum_{n=1}^{\lambda}\left[a_n\int_{-\pi}^{\pi} f(x)\cos nx + b_n\int_{-\pi}^{\pi} f(x)\sin nx\right]dx \geq 0$$

$$\Rightarrow \int_{-\pi}^{\pi} f^2(x)\,dx + \lim_{\lambda\to\infty}\sum_{n=1}^{\infty}\left[\pi a_n^2 + \pi b_n^2 - 2\pi a_n^2 - 2\pi b_n^2\right] \geq 0$$

$$\Rightarrow \int_{-\pi}^{\pi} f^2(x)\,dx - \pi\lim_{\lambda\to\infty}\sum_{n=1}^{\lambda}\left[a_n^2 + b_n^2\right] \geq 0$$

$$\Rightarrow \int_{-\pi}^{\pi} f^2(x)\,dx \geq \pi\lim_{\lambda\to\infty}\sum_{n=1}^{\lambda}\left[a_n^2 + b_n^2\right]$$

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